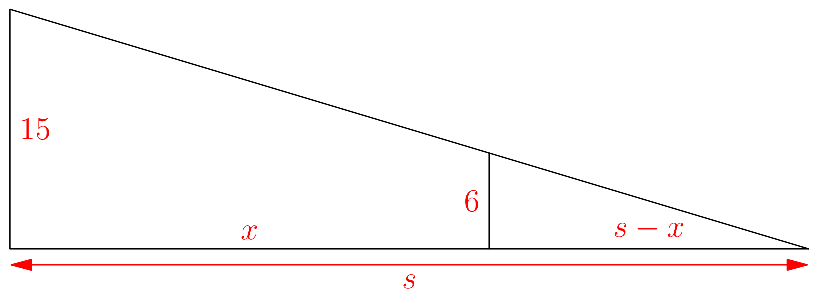
Math 103 Selected Homework Solutions, Section 3.9

9. Let s be the distance from the base of the light pole to the top of the man's shadow, and x the distance from the light pole to the man.



We know: $\frac{dx}{dt} = 5$ ft/s

We want to find: $\frac{ds}{dt}$ when $x = 40$ ft

Connecting equation: Similar triangles give us $\dfrac{s}{15} = \dfrac{s-x}{6}$. We can rewrite this as $6s = 15s - 15x$, or $5x = 3s$.

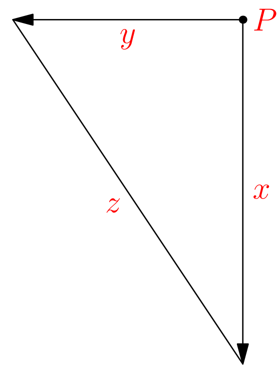
Differentiate: $5\frac{dx}{dt} = 3\frac{ds}{dt}$

Substitute and solve: It turns out that we can solve for $\frac{ds}{dt}$ by simply knowing $\frac{dx}{dt}$ (the information that $x = 40$ ft is irrelevant). Then:

$$5\frac{dx}{dt} = 3\frac{ds}{dt}$$
$$5(5) = 3\frac{ds}{dt}$$
$$\frac{ds}{dt} = \frac{25}{3}$$

So the tip of the man's shadow is moving at $\frac{25}{3}$ feet per second.

11. Let P be the starting point of the cars. Let x be the distance from P to the car that goes south, and y be the distance from P to the car that goes west. Let z be the distance between the two cars.



We know: $\frac{dx}{dt} = 60$ mi/h and $\frac{dy}{dt} = 25$ mi/h.

We want to find: $\frac{dz}{dt}$ when $t = 2$ hours.

Connecting equation: $x^2 + y^2 = z^2$ (Pythagorean theorem)

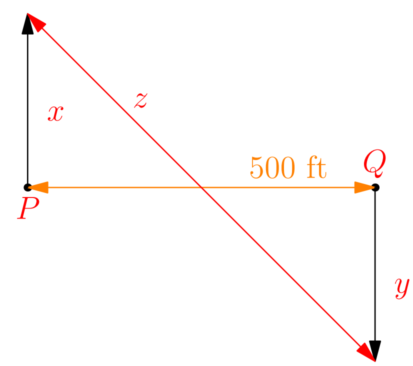
Differentiate: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$, or equivalently $x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$.

Substitute and solve: Two hours after the cars started moving, $x = 120$ miles, $y = 50$ miles, and $z = 130$ miles. Thus, we have:

$$\begin{aligned} x\frac{dx}{dt} + y\frac{dy}{dt} &= z\frac{dz}{dt} \\ (120)(60) + (50)(25) &= (130)\frac{dz}{dt} \\ \frac{dz}{dt} &= 65 \text{ mi/h} \end{aligned}$$

So the distance between the cars is increasing at 65 miles per hour.

13. Suppose the man starts walking north from P at $t = 0$ min, and the woman starts walking south from Q, 500 feet to the east of P, at $t = 5$ min. Let x be the distance from point P to the man, and let y be the distance from Q to the woman. Let z be the distance from the man to the woman, as in the diagram:



We know: $\frac{dx}{dt} = 4$ ft/s and $\frac{dy}{dt} = 5$ ft/s.

We want to find: $\frac{dz}{dt}$ when $t = 20$ min

Connecting equation: By the Pythagorean theorem, $(x+y)^2 + 500^2 = z^2$, or equivalently $x^2 + 2xy + y^2 + 500^2 = z^2$.

Differentiate: $2x\frac{dx}{dt} + 2\left(\frac{dx}{dt}y + x\frac{dy}{dt}\right) 2y\frac{dy}{dy} = 2z\frac{dz}{dt}$, which we can simplify to:

$$x\frac{dx}{dt} + \left(\frac{dx}{dt}y + x\frac{dy}{dt}\right) + y\frac{dy}{dy} = z\frac{dz}{dt}$$

or simplify further to:

$$(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = z\frac{dz}{dt}$$

Substitute and solve: At $t = 20$ min, $x = 4800$ ft, $y = 4500$ ft, and $z = 100\sqrt{8674}$. Thus, we have:

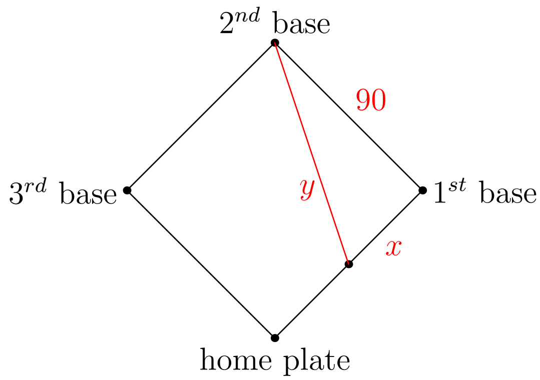
$$(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) = z\frac{dz}{dt}$$

$$(4800 + 4500)(4+5) = 100\sqrt{8674}\frac{dz}{dt}$$

$$\frac{837}{\sqrt{8674}} = \frac{dz}{dt}$$

So the distance is increasing at $\frac{837}{\sqrt{8674}}$, or approximately 8.99, feet per second.

14. (a) Let x be the distance from the runner to first base, and let y be the distance from the runner to second base.



We know: $\frac{dx}{dt} = -24$ ft/s (distance from runner to first base is *decreasing*)

We want to find: $\frac{dy}{dt}$ when $x = 45$ feet

Connecting equation: $x^2 + 90^2 = y^2$ (Pythagorean theorem)

Differentiate: $2x\frac{dx}{dt} + 0 = 2y\frac{dy}{dt}$, or equivalently $x\frac{dx}{dt} = y\frac{dy}{dt}$.

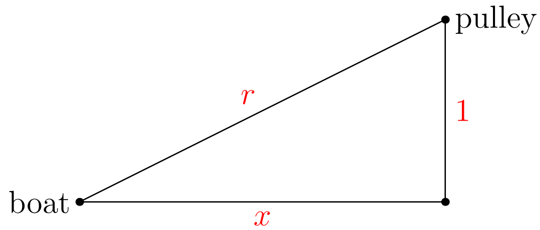
Substitute and solve: When $x = 45$, $y = 45\sqrt{5}$. Therefore,

$$\begin{aligned} x\frac{dx}{dt} &= y\frac{dy}{dt} \\ (45)(-24) &= (45\sqrt{5})\frac{dy}{dt} \\ \frac{dy}{dt} &= \frac{-24}{\sqrt{5}} \approx -10.7 \end{aligned}$$

So the distance from the runner to second base is decreasing at approximately 10.7 feet per second.

Part (b) is very similar, except it uses a different triangle in the diagram. However, the answer is evident by symmetry. When the runner is halfway between home plate and first base, his distance from second base is increasing at exactly the rate that his distance from third base is decreasing. Thus, when the runner is halfway to first base, his distance from third base is increasing at 10.73 feet per second.

16. Let r be the length of the rope between the boat and the pulley, and x be the distance from the boat to the dock, as in the diagram.



We know: $\frac{dr}{dt} = -1$ m/s

We want to find: $\frac{dx}{dt}$ when $x = 8$ m

Connecting equation: $x^2 + 1^2 = r^2$ (Pythagorean theorem)

Differentiate: $2x\frac{dx}{dt} + 0 = 2r\frac{dr}{dt}$, or equivalently $x\frac{dx}{dt} = r\frac{dr}{dt}$.

Substitute and solve: When $x = 8$, $r = \sqrt{65}$,

$$\begin{aligned} x\frac{dx}{dt} &= r\frac{dr}{dt} \\ (8)\frac{dx}{dt} &= (\sqrt{65})(-1) \\ \frac{dx}{dt} &= \frac{-\sqrt{65}}{8} \approx -1.007 \text{ m/s} \end{aligned}$$

So the boat is approaching the dock at approximately 1.007 meters per second.

19. Let V be the volume of the water in the tank (cm^3), and h be the depth of the water in the tank (cm). Let R be the rate at which water is being pumped into the tank (cm^3/min).

We know: $\frac{dV}{dt} = -10,000 + R$ and $\frac{dh}{dt} = 20$ when $h = 200$.

We want to find: R

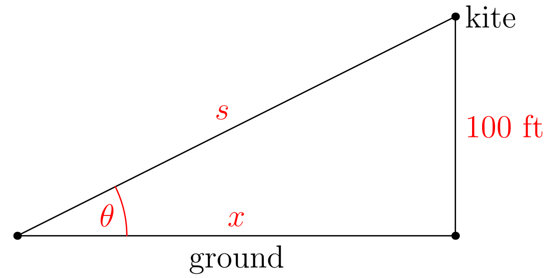
Connecting equation: The volume of a cone is
$V = \frac{1}{3}$(area of base)(height) $= \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h$, so $V = \frac{\pi}{27}h^3$.

Differentiate: $\frac{dV}{dt} = \frac{\pi}{9}h^2\frac{dh}{dt}$

Substitute and solve:

$$\begin{aligned}\frac{dV}{dt} &= \frac{\pi}{9}h^2\frac{dh}{dt}\\ -10,000 + R &= \frac{\pi}{9}(200)^2(20)\\ R &= 10,000 + 800,000\frac{\pi}{9}\\ R &\approx 289,000 \text{ cm}^3/\text{min}\end{aligned}$$

24. Let s be the length of the kite string, x the distance from the person holding the string to the spot on the ground directly below the kite, and θ the angle between the string and the horizontal.



We know: $\frac{dx}{dt} = 8$ ft/s

We want to find: $\frac{d\theta}{dt}$ when $s = 200$ ft; that is, when $x = 100\sqrt{3}$

Connecting equation: $\tan\theta = \dfrac{100}{x}$

Differentiate: $\sec^2(\theta)\frac{d\theta}{dt} = -100x^{-2}\frac{dx}{dt}$

Substitute and solve: When $x = 100\sqrt{3}$, $\theta = \frac{\pi}{6}$, and we have:

$$\begin{aligned}
\sec^2(\theta)\frac{d\theta}{dt} &= -100x^{-2}\frac{dx}{dt} \\
\sec^2\left(\frac{\pi}{6}\right)\frac{d\theta}{dt} &= -100(100\sqrt{3})^{-2}(8) \\
\frac{4}{3}\frac{d\theta}{dt} &= \frac{-2}{75} \\
\frac{d\theta}{dt} &= \frac{-1}{50} = -0.02
\end{aligned}$$

So the angle is decreasing at 0.02 radians per second.